The transverse force on a spinning sphere moving in a viscous fluid

By S. I. RUBINOW
Stevens Institute of Technology, Hoboken, New Jersey
AND JOSEPH B. KELLER
Institute of Mathematical Sciences, New York University, New York, N.Y.

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The flow about a spinning sphere moving in a viscous fluid is calculated for small values of the Reynolds number. With this solution the force and torque on the sphere are computed. It is found that in addition to the drag force determined by Stokes, the sphere experiences a force $\mathbf{F}_L$ orthogonal to its direction of motion. This force is given by

$$\mathbf{F}_L = \pi a^3 \rho \mathbf{\Omega} \times \mathbf{V}[1+O(R)].$$

Here a is the radius of the sphere, $\mathbf{\Omega}$ is its angular velocity, $\mathbf{V}$ is its velocity, ρ is the fluid density and R is the Reynolds number, $R = \rho\mu^{-1}Va$. For small values of R, the transverse force is independent of the viscosity μ. This force is in such a direction as to account for the curving of a pitched baseball, the long range of a spinning golf ball, etc. It is used as a basis for the discussion of the flow of a suspension of spheres through a tube.

The calculation involves the Stokes and Oseen expansions. A representation of solutions of the Oseen equations in terms of two scalar functions is also presented.

1. Introduction

We shall show that a spinning sphere moving in a viscous fluid experiences a force orthogonal to its direction of motion, which we call a lift force. It is also orthogonal to the spin axis and therefore accounts for the curving of a pitched baseball, the long range of a spinning golf ball, etc. Since we shall calculate the lift for small values of the Reynolds number, our result will not apply to these phenomena, which occur for large Reynolds numbers. Our result applies either to the slow motion of a small sphere in a fluid, such as a solid particle in a suspension, or to the motion of a satellite or other astronomical object in a gas of low density. We shall also show that the spin does not effect the drag force and that there is no correction to the retarding torque to the order we consider. Using these results we shall determine the motion of a sphere with given initial linear and angular velocities. We shall also consider the flow of a suspension of spheres through a tube, in the light of our results.

Our results are obtained by solving the Navier–Stokes equations for the motion of the fluid around the sphere. We determine the first few terms, in the expansion in terms of the Reynolds number, of the solution. This expansion consists of two parts, the Stokes and Oseen expansions, which were introduced and used by Lagerstrom & Cole (1955), and Proudman & Pearson (1957). Our method of solution is similar to theirs with the main difference being that we obtain directly the expansion for the velocity and the pressure rather than for the stream function. This is necessary because our flow is asymmetric and therefore there is no stream function which describes it. Those authors, as well as Stokes (1851) and Oseen (1910, 1913) considered the motion of a non-spinning sphere, about which the flow is axially symmetric and describable by a stream function. The present example appears to be the first asymmetric three-dimensional case for which the flow at small Reynolds numbers has been found by using these expansions. In the appendix we prove that it is possible to represent any solution of the Oseen equations in terms of two scalar functions.

2. Formulation

We wish to consider the motion of a sphere of radius a, velocity $\mathbf{V}$ and angular velocity $\mathbf{\Omega}$ through an incompressible viscous fluid of density ρ and viscosity coefficient μ. The fluid is assumed to be at rest far from the sphere. It is convenient to introduce a co-ordinate system with its origin at the centre of the sphere, with the negative x'-axis pointing in the direction of $\mathbf{V}$ and with the (x', y')-plane containing $\mathbf{\Omega}$. In this co-ordinate system the flow is steady and the fluid has the velocity $(V, 0, 0)$ at infinity while the velocity of the surface of the sphere $r' = a$ is $\mathbf{\Omega} \times \mathbf{r}'$. We denote the velocity in the fluid by $\mathbf{u}'$ and the pressure in the fluid by p', taking the pressure at infinity to be zero. Now $\mathbf{u}'$ and p' satisfy the time independent Navier–Stokes equations, the equation of continuity, the condition of no slip at the sphere and the appropriate conditions at infinity. These equations and conditions are

$$\mu\Delta\mathbf{u}' - \nabla p' = \rho(\mathbf{u}'.\nabla)\mathbf{u}', \tag{1}$$

$$\nabla.\mathbf{u}' = 0, \tag{2}$$

$$\mathbf{u}' = \mathbf{\Omega}\times\mathbf{r}' \quad \text{at} \quad r' = a, \tag{3}$$

$$\mathbf{u}' = (V, 0, 0) \quad \text{at} \quad r' = \infty, \tag{4}$$

and

$$p' = 0 \quad \text{at} \quad r' = \infty. \tag{5}$$

Let us introduce the dimensionless Reynolds number R as well as the dimensionless quantities $\mathbf{u}$, p, $\boldsymbol{\omega}$, x, y, z and $\mathbf{r}$, defined by

$$\left.\begin{aligned} x &= a^{-1}x', & y &= a^{-1}y', & z &= a^{-1}z', & \mathbf{r} &= a^{-1}\mathbf{r}', \\ p &= aV^{-1}\mu^{-1}p', & \mathbf{u} &= V^{-1}\mathbf{u}', & \boldsymbol{\omega} &= aV^{-1}\mathbf{\Omega}, & R &= \rho\mu^{-1}Va. \end{aligned}\right\} \tag{6}$$

Upon introducing these variables into (1)–(5), we obtain

$$\Delta\mathbf{u} - \nabla p = R(\mathbf{u}.\nabla)\mathbf{u}, \tag{7}$$

$$\nabla.\mathbf{u} = 0, \tag{8}$$

$$\mathbf{u} = \boldsymbol{\omega}\times\mathbf{r} \quad \text{at} \quad r = 1, \tag{9}$$

$$\mathbf{u} = (1, 0, 0) \quad \text{at} \quad r = \infty, \tag{10}$$

and

$$p = 0 \quad \text{at} \quad r = \infty. \tag{11}$$

We seek the solution $\mathbf{u}$ and p of (7)–(11) as expansions in R valid for small values of R.

3. Stokes expansion

The Stokes expansion of the solution is of the form

$$\mathbf{u} = \mathbf{u}_0 + R\mathbf{u}_1 + o(R), \tag{12}$$

$$p = p_0 + Rp_1 + o(R). \tag{13}$$

Further terms beyond those shown involve powers of $\log R$ in addition to powers of R, as has been demonstrated by Proudman & Pearson (1957) for the non-rotating sphere. However, only the terms shown in (12) and (13) will be needed here. We now insert (12) and (13) into (7)–(9) and equate coefficients of R^0 in each equation, obtaining

$$\Delta \mathbf{u}_0 - \nabla p_0 = 0, \tag{14}$$

$$\nabla . \mathbf{u}_0 = 0, \tag{15}$$

and

$$\mathbf{u}_0 = \boldsymbol{\omega} \times \mathbf{r} \quad \text{at} \quad r = 1. \tag{16}$$

From the coefficients of R^1, we obtain

$$\Delta \mathbf{u}_1 - \nabla p_1 = (\mathbf{u}_0 . \nabla)\, \mathbf{u}_0, \tag{17}$$

$$\nabla . \mathbf{u}_1 = 0, \tag{18}$$

and

$$\mathbf{u}_1 = 0 \quad \text{at} \quad r = 1. \tag{19}$$

We have not required of the Stokes expansion that it satisfy (10) and (11), the conditions at infinity, because this expansion is not uniformly valid in the neighbourhood of infinity. Therefore conditions at infinity on the individual terms cannot be obtained in the same way. Instead we must obtain another expansion, the Oseen expansion, which is valid at infinity. Then by matching the two expansions the necessary conditions will be obtained. To obtain the Oseen expansion we must first introduce stretched variables.

4. Stretched variables

Let us introduce the new stretched variables X, Y, Z and $\mathbf{s}$ and the new functions $\mathbf{U}$ and P, defined by

$$\left.\begin{aligned} X = Rx, \quad Y = Ry, \quad Z = Rz, \quad \mathbf{s} = R\mathbf{r}, \\ \mathbf{U}(X, Y, Z, R) = \mathbf{u}(R^{-1}X, R^{-1}Y, R^{-1}Z, R), \\ P(X, Y, Z, R) = R^{-1}p(R^{-1}X, R^{-1}Y, R^{-1}Z, R). \end{aligned}\right\} \tag{20}$$

In terms of these variables, equations (7)–(11) become

$$\Delta \mathbf{U} - \nabla P = (\mathbf{U} . \nabla)\, \mathbf{U}, \tag{21}$$

$$\nabla . \mathbf{U} = 0, \tag{22}$$

$$\mathbf{U} = R^{-1}\boldsymbol{\omega} \times \mathbf{s} \quad \text{at} \quad s = R, \tag{23}$$

$$\mathbf{U} = (1, 0, 0) \quad \text{at} \quad s = \infty, \tag{24}$$

and

$$P = 0 \quad \text{at} \quad s = \infty. \tag{25}$$

5. Oseen expansion

The Oseen expansion of the solution $\mathbf{U}$ and P of (21)–(25) is of the form

$$\mathbf{U} = \mathbf{U}_0 + R\mathbf{U}_1 + o(R), \tag{26}$$

$$P = P_0 + RP_1 + o(R). \tag{27}$$

Upon inserting (26) and (27) into (21), (22), (24) and (25) and equating coefficients of R^0, we obtain

$$\Delta \mathbf{U}_0 - \nabla P_0 = (\mathbf{U}_0 . \nabla)\, \mathbf{U}_0, \tag{28}$$

$$\nabla . \mathbf{U}_0 = 0, \tag{29}$$

$$\mathbf{U}_0 = (1, 0, 0) \quad \text{at} \quad s = \infty, \tag{30}$$

and

$$P_0 = 0 \qquad \text{at} \quad s = \infty. \tag{31}$$

Equating coefficients of R^1 yields the equations

$$\Delta \mathbf{U}_1 - \nabla P_1 = (\mathbf{U}_0 . \nabla)\, \mathbf{U}_1 + (\mathbf{U}_1 . \nabla)\, \mathbf{U}_0, \tag{32}$$

$$\nabla . \mathbf{U}_1 = 0, \tag{33}$$

$$\mathbf{U}_1 = (0, 0, 0) \quad \text{at} \quad s = \infty, \tag{34}$$

and

$$P_1 = 0 \qquad \text{at} \quad s = \infty. \tag{35}$$

We have not inserted the Oseen expansion into the condition (23) on the sphere because we do not expect this expansion to be uniformly valid there. Consequently this condition must be replaced by another condition. This latter condition is that the Stokes and Oseen expansions must match since they are both expansions of the same solution expressed in different variables. This matching will be explained later.

We note that equations (28)–(31) satisfied by $\mathbf{U}_0$ and P_0 are the same as the corresponding equations for $\mathbf{U}$ and P. Therefore we cannot expect to solve them more readily than we could solve the original equations. However, we observe that a particular solution for $\mathbf{U}_0$ and P_0 is

$$\mathbf{U}_0 = (1, 0, 0), \tag{36}$$

$$P_0 = 0. \tag{37}$$

The Oseen expansion is based upon choosing this particular solution for $\mathbf{U}_0$ and P_0. With this choice (32) simplifies to

$$\Delta \mathbf{U}_1 - \nabla P_1 = \frac{\partial \mathbf{U}_1}{\partial X}. \tag{38}$$

Before determining $\mathbf{U}_1$ and P_1, let us explain the principle of matching.

6. Matching

The matching principle is that two different asymptotic expansions of a given function must be asymptotically equal in their common domain of validity, if any. We assume that the Stokes expansion is valid from the sphere out to some large distance. We also assume that the Oseen expansion is valid from infinity in to some small radius in the stretched variables. However, this radius is considered to be large in the unstretched variables. Then the common domain of validity is a spherical shell within which the unstretched radius is large but the stretched radius is small. Therefore if the Stokes expansion is expanded for large values of the radius r while the Oseen expansion is expanded for small values of the stretched radius s, the resulting expansions must be asymptotically equal. This principle is utilized by alternately determining terms in the two expansions and matching them to the previous terms.

As a first result of applying this principle we can determine the behaviour of the zeroth-order Stokes approximation, $\mathbf{u}_0$ and p_0, for large values of r. Since $\mathbf{U}_0$ and P_0 are constant, the result is easily seen to be

$$\mathbf{u}_0 = \mathbf{U}_0 + o(1) \quad \text{as} \quad r \to \infty, \tag{39}$$

and

$$p_0 = o(1) \quad \text{as} \quad r \to \infty. \tag{40}$$

7. Zeroth-order Stokes approximation

The solution $\mathbf{u}_0$ and p_0 of (14)–(16), (39) and (40) is unique and can be found by the method explained by Lamb (1945, pp. 595, 596). Since the equations are all linear, the solution is just the sum of the known solution for a uniform flow past a non-rotating sphere and the known flow produced by a rotating sphere in a fluid at rest at infinity. Thus we have

$$\mathbf{u}_0 = \left(1 - \frac{3}{4r} - \frac{1}{4r^3}\right)(1,0,0) - \frac{3x}{4}\left(\frac{1}{r^3} - \frac{1}{r^5}\right)\mathbf{r} + \frac{1}{r^3}\boldsymbol{\omega} \times \mathbf{r}, \tag{41}$$

$$p_0 = -\frac{3x}{2r^3}. \tag{42}$$

8. First-order Oseen approximation

To determine $\mathbf{U}_1$ and P_1 we must solve (38), (33)–(35) and employ the matching condition. We begin by taking the divergence of (38) and using (33), from which it follows that $\Delta P_1 = 0$. Then, following Lamb (1945, p. 610), we introduce a scalar ϕ and a vector $\mathbf{W}$ and write

$$P_1 = \frac{\partial \phi}{\partial X}, \tag{43}$$

$$\mathbf{U}_1 = -\nabla\phi + \mathbf{W}. \tag{44}$$

Since P_1 is harmonic we require that ϕ also be harmonic, i.e.

$$\Delta\phi = 0. \tag{45}$$

Then (38) and (33) become two equations for $\mathbf{W}$:

$$\Delta \mathbf{W} - \frac{\partial}{\partial X}\mathbf{W} = 0, \tag{46}$$

$$\nabla . \mathbf{W} = 0. \tag{47}$$

In the appendix we prove the following theorem.

Theorem. If $\mathbf{W}$ satisfies (46) and (47) then there exist two scalar functions ψ and χ such that

$$\mathbf{W} = (1,0,0)\times\nabla\psi+\nabla\times[(1,0,0)\times\nabla\chi] \tag{48}$$

and

$$\Delta\psi-\frac{\partial}{\partial X}\psi = 0, \tag{49}$$

$$\Delta\chi-\frac{\partial}{\partial X}\chi = 0. \tag{50}$$

Let us define χ', which also satisfies (50), by

$$\chi' = \frac{\partial}{\partial X}\chi. \tag{51}$$

Then (48) can be written as

$$\mathbf{W} = \nabla\chi'+\left(-\chi', -\frac{\partial\psi}{\partial Z}, \frac{\partial\psi}{\partial Y}\right). \tag{52}$$

Those special solutions of the form (48) or (52) for which $\psi = 0$ were given by Lamb (1945, p. 611).

To construct $\mathbf{U}_1$ and P_1 let us make the following choices of ϕ, ψ and χ', which must be solutions of (45), (49) and (50) respectively:

$$\phi = A/s, \tag{53}$$

$$\psi = 0, \tag{54}$$

and

$$\chi' = (B/s)\exp\left[\tfrac{1}{2}(X-s)\right]. \tag{55}$$

We now compute $\mathbf{U}_1$ and P_1 from (43) and (44), using (52)–(55) and employ the matching condition. This yields $A = B = \frac{3}{2}$. Thus $\mathbf{U}_1$ and P_1 are found to be

$$\mathbf{U}_1 = \frac{3}{2s^3}(X,Y,Z)-\frac{3}{4s}\exp\left[\tfrac{1}{2}(X-s)\right](1,0,0)-\tfrac{3}{4}\exp\left[\tfrac{1}{2}(X-s)\right]\left(\frac{1}{s^2}-\frac{2}{s^3}\right)(X,Y,Z) \tag{56}$$

and

$$P_1 = -\frac{3X}{2s^3}. \tag{57}$$

This solution satisfies the condition at infinity (34) and (35). If any other choice were made for ϕ, ψ and χ', the matching condition or the conditions at infinity would have been violated.

9. First-order Stokes approximation

To determine $\mathbf{u}_1$ and p_1 we must solve (17)–(19) and employ the matching principle. We shall first find a particular solution $\mathbf{u}_{10}$ and p_{10} of (17) and (18). Then we shall add a solution $\mathbf{u}_{11}$ and p_{11} of (18) and the homogeneous form of (17) in order to satisfy the boundary condition (19). Finally we shall show that the sum of the two solutions so obtained satisfies the matching condition so no other solution need be added. To begin, we compute the inhomogeneous term $(\mathbf{u}_0.\nabla)\,\mathbf{u}_0$ which occurs in (17). By using (41) for $\mathbf{u}_0$ we obtain

$$\begin{aligned}(\mathbf{u}_0.\nabla)\,\mathbf{u}_0 = {}& \frac{3x}{16}\left[-\frac{3}{r^4}+\frac{8}{r^5}-\frac{6}{r^6}+\frac{1}{r^8}\right](1,0,0)+\frac{3}{16}\left[-\frac{4}{r^3}+\frac{3}{r^4}+\frac{4}{r^5}-\frac{2}{r^6}-\frac{1}{r^8}\right.\\ &\left.+x^2\left(\frac{12}{r^5}-\frac{12}{r^6}-\frac{20}{r^7}+\frac{24}{r^8}-\frac{4}{r^{10}}\right)\right]\mathbf{r}-3x\left(\frac{1}{r^5}-\frac{1}{r^6}\right)\boldsymbol{\omega}\times\mathbf{r}\\ &-\frac{3\omega z\sin\beta}{4}\left(\frac{1}{r^6}-\frac{1}{r^8}\right)\mathbf{r}-\frac{\omega\sin\beta}{4}\left[\frac{4}{r^3}-\frac{3}{r^4}-\frac{1}{r^6}\right](0,0,1)+\frac{1}{r^6}\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}).\end{aligned} \tag{58}$$

Here β is the angle between $\boldsymbol{\Omega}$ and $\mathbf{U}_0$.

Let us now take the divergence of (17) and make use of (18) and (58). Then we find that

$$\Delta p_1 = -\nabla . [(\mathbf{u}_0 . \nabla)\,\mathbf{u}_0] = \frac{9}{8}\left[-\frac{1}{r^4}+\frac{1}{r^8}+x^2\left(\frac{4}{r^6}-\frac{6}{r^8}+\frac{2}{r^{10}}\right)\right]$$
$$-\frac{3\omega z \sin\beta}{2}\left(\frac{1}{r^6}-\frac{1}{r^8}\right)+\frac{4\omega^2}{r^6}-\frac{(\boldsymbol{\omega}.\mathbf{r})^2}{r^8}. \qquad (59)$$

A particular solution of (59) is

$$p_{10} = \frac{3}{32}\left[-\frac{6}{r^2}-\frac{2}{r^4}-\frac{1}{3r^6}+x^2\left(\frac{12}{r^4}+\frac{12}{r^6}-\frac{1}{r^8}\right)\right]-\frac{\omega z \sin\beta}{8}\left(\frac{3}{r^4}-\frac{2}{r^6}\right)-\frac{\omega^2}{2r^4}+\frac{(\boldsymbol{\omega}.\mathbf{r})^2}{r^6}. \qquad (60)$$

We now insert (60) into (17) and obtain for $\mathbf{u}_1$ the equation

$$\Delta \mathbf{u}_1 = \nabla p_{10} + (\mathbf{u}_0 . \nabla)\,\mathbf{u}_0. \qquad (61)$$

A particular solution of (61) is

$$\mathbf{u}_{10} = \frac{3}{32}\left[4-x\left(\frac{4}{r}-\frac{3}{r^2}+\frac{1}{r^4}\right)\right](1,0,0)+\frac{3}{32}\left[\frac{2}{r}-\frac{3}{r^2}-\frac{1}{r^4}\right.$$
$$\left.+x^2\left(-\frac{2}{x^3}+\frac{6}{r^4}+\frac{4}{r^5}+\frac{4}{r^6}\right)\right]\mathbf{r}+\frac{x}{4}(\boldsymbol{\omega}\times\mathbf{r})\left(\frac{2}{r^3}-\frac{3}{r^4}\right)$$
$$+\frac{\omega\sin\beta}{16}\left[\frac{8}{r}-\frac{6}{r^2}+\frac{1}{r^4}\right](0,0,1)-\frac{\omega z\sin\beta}{16}\left[\frac{3}{r^4}+\frac{2}{r^6}\right]\mathbf{r}$$
$$-\frac{\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})}{4r^4}-\frac{\omega^2}{2r^4}\mathbf{r}+\frac{(\boldsymbol{\omega}.\mathbf{r})^2}{r^6}\mathbf{r}. \qquad (62)$$

By direct computation we find that $\mathbf{u}_{10}$ satisfies (18).

Next we find $\mathbf{u}_{11}$ and p_{11} satisfying the homogeneous form of (17) and (18) and such that $\mathbf{u}_{10}+\mathbf{u}_{11}$ satisfies (19). Thus we require that

$$\mathbf{u}_{11} = -\mathbf{u}_{10} \quad \text{at} \quad r = 1. \qquad (63)$$

By the method of Lamb (1945, pp. 595, 596), we find that

$$\mathbf{u}_{11} = \frac{3}{32}\left[-\frac{3}{r}-\frac{1}{r^3}+\frac{2x}{r^5}\right](1,0,0)+\frac{3}{32}\left[\frac{1}{r^3}+\frac{1}{r^5}-3x\left(\frac{1}{r^3}-\frac{1}{r^5}\right)-x^2\left(\frac{7}{r^5}+\frac{5}{r^7}\right)\right]\mathbf{r}$$
$$+\frac{x}{4r^5}\,\boldsymbol{\omega}\times\mathbf{r}-\frac{\omega\sin\beta}{16}\left(\frac{2}{r}+\frac{1}{r^3}\right)(0,0,1)-\frac{\omega z\sin\beta}{16}\left(\frac{2}{r^3}-\frac{7}{r^5}\right)\mathbf{r}$$
$$+\frac{\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})}{4r^5}+\frac{\omega^2}{8}\left(\frac{1}{r^3}+\frac{3}{r^5}\right)\mathbf{r}-\frac{(\boldsymbol{\omega}.\mathbf{r})^2}{8}\left[\frac{3}{r^5}+\frac{5}{r^7}\right]\mathbf{r}, \qquad (64)$$

and

$$p_{11} = -\frac{9}{16}\frac{x}{r^3}+\frac{7}{16}\left(\frac{1}{r^3}-\frac{3x^2}{r^5}\right)-\frac{\omega z\sin\beta}{4r^3}+\frac{\omega^2}{4r^3}-\frac{3(\boldsymbol{\omega}.\mathbf{r})^2}{4r^5}. \qquad (65)$$

Upon adding $\mathbf{u}_{10}$ to $\mathbf{u}_{11}$ and p_{10} to p_{11}, we obtain $\mathbf{u}_1$ and p_1. (There is no need here to write out the resulting expressions for $\mathbf{u}_1$ and p_1, but we shall presently refer to them as (66) and (67), respectively.) By employing the matching procedure we find that this solution matches the Oseen solution, and therefore it is not necessary to add anything else to it.

10. Lift, drag and torque

We shall now use the two terms of the Stokes expansion which we have found to compute the force and torque on the sphere. To do so we must first determine **f**, the force per unit area exerted by the fluid on the surface of the sphere. Lamb (1945, p. 596) gives an expression for **f** which becomes in dimensionless variables

$$\mathbf{f} = \frac{V\mu}{a}\left[-\frac{\mathbf{r}}{r}p + \left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\mathbf{u} + \frac{1}{r}\nabla(\mathbf{r}.\mathbf{u})\right]. \tag{68}$$

Upon using (41), (42), (66) and (67) in (68) and integrating (68) over the surface of the sphere we obtain the total force **F** which is given in terms of the velocity **V** of the sphere by

$$\mathbf{F} = -6\pi a\mu\mathbf{V}(1+\tfrac{3}{8}R) + \pi a\mu R\boldsymbol{\omega}\times\mathbf{V} + o(a\mu VR). \tag{69}$$

The component of **F** in the direction $-\mathbf{V}$ is the drag force $\mathbf{F}_D$ and the component normal to **V** is the lift force $\mathbf{F}_L$. From (69) and the definition (6) of R, these components are

$$\mathbf{F}_D = -6\pi a\mu\mathbf{V}[1+\tfrac{3}{8}R + o(R)] \tag{70}$$

and

$$\mathbf{F}_L = \pi a^3\rho\boldsymbol{\Omega}\times\mathbf{V}[1+O(R)]. \tag{71}$$

It is interesting to note that the leading term in $\mathbf{F}_L$ is independent of μ and is similar to the lift formula for two-dimensional potential flow about an airfoil. By integrating $\mathbf{r}\times\mathbf{f}$ over the surface of the sphere we obtain for the torque **T** on the sphere the result

$$\mathbf{T} = -8\pi\mu a^3\boldsymbol{\Omega}[1+o(R)]. \tag{72}$$

The term $-6\pi a\mu\mathbf{V}$ in $\mathbf{F}_D$ was calculated by Stokes (1861) and the second term $-\frac{9}{4}\pi\mu a VR$ by Oseen (1910, 1913). The expression for $\mathbf{F}_L$, which is new, is the main result of this paper. The term $-8\pi\mu a^3\boldsymbol{\Omega}$ in **T** was obtained by Kirchhoff (1876). Our results show that the spin causes no correction to the drag of order R and that there is no correction to the torque of order R.

11. The motion of a spinning sphere

Let us now consider the motion of a sphere with initial velocity $\mathbf{V}_0$ and initial angular velocity $\boldsymbol{\Omega}_0$ in a viscous fluid initially at rest. Let M denote the mass of the sphere, I its moment of inertia about any diameter, $\mathbf{V}(t)$ its linear velocity and $\boldsymbol{\Omega}(t)$ its angular velocity about a diameter at time t. We assume that the force and torque on the sphere are given by (69) and (72), ignoring the terms $o(a\mu VR)$ in (69) and $o(R)$ in (72). Then the equations of motion of the sphere are

$$M\frac{d\mathbf{V}}{dt} = -6\pi a\mu(1+\tfrac{3}{8}R)\mathbf{V} + \pi a^3\rho\boldsymbol{\Omega}\times\mathbf{V} \tag{73}$$

and

$$I\frac{d\boldsymbol{\Omega}}{dt} = -8\pi\mu a^3\boldsymbol{\Omega}. \tag{74}$$

The solution of (74) is

$$\boldsymbol{\Omega}(t) = \boldsymbol{\Omega}_0\exp\left[-\frac{8\pi\mu a^3}{I}t\right]. \tag{75}$$

If $\mathbf{V}^{(1)}(t)$ denotes the component of $\mathbf{V}(t)$ parallel to $\mathbf{\Omega}_0$, (73) yields

$$\mathbf{V}^{(1)}(t) = \mathbf{V}_0^{(1)} \exp\left[-6\pi a\mu(1+\tfrac{3}{8}R)\, M^{-1}t\right]. \tag{76}$$

Let $\mathbf{V}^{(2)}(t)$ denote the component of $\mathbf{V}(t)$ perpendicular to $\mathbf{\Omega}_0$. Then (73) yields

$$|\mathbf{V}^{(2)}(t)| = |\mathbf{V}_0^{(2)}| \exp\left[-6\pi a\mu(1+\tfrac{3}{8}R)\, M^{-1}t\right]. \tag{77}$$

Now we define the unit vector $\mathbf{v}(t)$ by

$$\mathbf{V}^{(2)}(t) = |\mathbf{V}^{(2)}(t)|\, \mathbf{v}(t). \tag{78}$$

From (73) it follows that $\mathbf{v}(t)$ satisfies the equation

$$\frac{d\mathbf{v}}{dt} = \frac{\pi a^3\rho}{M} \exp\left[-\frac{8\pi\mu a^3 t}{I}\right] \mathbf{\Omega}_0 \times \mathbf{v}. \tag{79}$$

The solution of (79) is, in rectangular components,

$$v(t) = [\cos\theta(t), \sin\theta(t)]. \tag{80}$$

Here the angle $\theta(t)$ is given by

$$\theta(t) = \theta_0 + \frac{I\rho\,|\mathbf{\Omega}_0|}{8\mu M}\left(1 - \exp\left[-\frac{8\pi\mu a^3}{I} t\right]\right). \tag{81}$$

From these results the path of the sphere can be obtained easily by integration.

12. Flow of a suspension through a tube

When a viscous fluid containing suspended particles flows through a circular tube, the particles are not uniformly distributed over the cross-section of the tube. Observations on blood flow in narrow capillaries indicate that the red blood corpuscles concentrate near the axis of the capillary. This concentration effect was explained hydrodynamically by supposing that the fluid exerted an axially directed transverse force on the particles. However, when the flow was determined by solving the linearized Stokes equations, no transverse force was found on a sphere in a shear flow, such as the Poiseuille flow in a circular tube.† A similar calculation of the force on an ellipsoid in a shear flow by Jeffery (1922) also yielded no transverse force.

To account for the effect, Jeffery proposed a new principle—that a particle moves at that distance from the axis which minimizes the rate of energy dissipation in the flow. This principle was subsequently rediscovered and generalized and has become a cornerstone of irreversible thermodynamics. If correct, Jeffery's principle should be deducible from the Navier–Stokes equations, just as all thermodynamic principles are deducible from more detailed theories. To apply it to the present problem, Jeffery utilized the result of Einstein (1906) for the excess dissipation produced by the introduction of a sphere into a uniform shear flow. Einstein has shown that the excess dissipation is proportional to

† This result can be obtained without calculation by decomposing the incident shear flow into two parts, one symmetric and the other antisymmetric about any plane through the centre of the sphere and containing the direction of the incident flow through the centre. By symmetry, neither of these flows can yield a force on the sphere normal to the plane.

the square of the vorticity of the incident flow at the particle. Since the velocity distribution in the Poiseuille flow is parabolic, the vorticity is proportional to the distance from the axis and the excess energy dissipation to the square of the distance from the axis. Thus Jeffery concluded that a particle would move along the axis.

This same conclusion was arrived at by Tollert (1954) and Saffman (1956). Tollert observed that, according to Einstein, a sphere in a shear flow $\mathbf{u}$ rotates with the angular velocity $\mathbf{\Omega} = \frac{1}{2}\nabla \times \mathbf{u}$. Furthermore, according to the calculation of Simha (1936), a sphere of radius a in a tube of radius R_0 lags behind the flow with the relative velocity $\mathbf{V} = -\frac{2}{3}(a^2/R_0^2)\,\mathbf{U}_0$, where $\mathbf{U}_0$ is the flow velocity at the centre of the tube. Then Tollert postulated a transverse force of the form (71) with another factor instead of $\frac{2}{3}\pi$. In a Poiseuille flow, $\nabla \times \mathbf{u} = -2\mathbf{b} \times \mathbf{U}_0/R_0^2$, where $\mathbf{b}$ denotes the radial vector from the axis. If the correct formula (71) for the transverse force on a sphere at radial position $\mathbf{b}$ were used, Tollert's argument would yield the result

$$\mathbf{F}_L = -2\pi a^5 \rho U_0^2 \mathbf{b}/3R_0^4. \tag{82}$$

Saffman derived a similar result by approximate solution of the flow equations, obtaining the numerical factor $7{\cdot}7(\frac{2}{3}\pi)$ instead of $\frac{2}{3}\pi$. The result (82) yields a force toward the axis which vanishes on the axis, in agreement with Jeffery's conclusion.

Recently Segre & Silberberg (1961) measured very accurately the distribution of spherical particles in a fluid flowing in a circular tube. They found that the particles do not concentrate along the axis. Instead they concentrate at the radius b_0 equal to about $\frac{1}{2}R_0$. Thus the previous observations were inaccurate, and the theories of Jeffery, Tollert and Saffman are incomplete. Consequently the formula (82) is incorrect, or at best incomplete. This formula and the theories referred to only account for the inward motion of spherical particles lying outside the radius b_0.

In order to understand why (82) fails, we must recognize that it is based upon (71), which applies to a sphere in a uniform flow. But the Poiseuille flow in a tube is not uniform. It is a shear flow with a parabolic velocity profile. The gradient of this profile accounts for the spin of the sphere and the curvature of the profile accounts for the lag. However, these are evidently not the only ways in which the non-uniformity of the flow affects the sphere. We therefore attempted to calculate the flow about a spinning sphere in a shear flow with a parabolic velocity profile, and from it to compute the force and torque on the sphere. We followed the procedure employed in the preceding sections and had no difficulty in obtaining the zero-order Stokes solution. We began to calculate the first-order Stokes solution and obtained some terms in it, but we did not complete the calculation because the labour became prohibitive. However, from the terms we calculated, we computed the transferse force, which came out to be given by (82). But there are many more terms in the force which we did not calculate. We expect that these further terms would modify (82) in such a way as to account for the observed effect, but this is merely a conjecture. This conjecture is strengthened by the fact that (82) yields a force of the correct order of magnitude to account for the observed effect, as we shall show below.

We have also examined the minimum-dissipation-rate analysis of Jeffery and found that it utilizes the linearized flow about a sphere in an unbounded uniform shear flow. But to zero order in the Reynolds number, the correct flow is the linearized flow about a sphere in a Poiseuille flow in a tube. If this zero-order flow were used in calculating the rate of energy dissipation, it might yield the correct equilibrium distance of the sphere from the axis.

In view of the experimental results, let us now modify (82) so that it gives an outward force for$b < b_0$. The simplest way to do this is to multiply the right-hand side of (82) by $(b-b_0)/b_0$. With this factor the force is inward for $b > b_0$ and outward for $b < b_0$. Of course this expression for the force is not correct in detail, but it is qualitatively correct.

We shall now calculate the trajectory of a sphere subject to this force. Its radial velocity $d\mathbf{b}/dt$ can be determined by equating the drag force $6\pi a\mu d\mathbf{b}/dt$ to $\mathbf{F}_L$. Thus

$$\frac{db}{dt} = -\frac{a^4\rho U_0^2}{9R_0^4\mu}\frac{b(b-b_0)}{b_0}. \tag{83}$$

Now the axial velocity of the sphere dx/dt is practically equal to the flow velocity $U_0(1-b^2R_0^{-2})$. Thus we have, upon dividing,

$$\frac{dx}{db} = -\frac{9R_0^2\mu b_0(R_0^2-b^2)}{a^4\rho U_0 b(b-b_0)}. \tag{84}$$

We now integrate (84) from $x = 0$, denoting the initial radius of the particle at $x = 0$ by b_1. Thus we obtain

$$x = \frac{9R_0^2\mu b_0}{a^4\rho U_0}\left[b-b_1+\frac{R_0^2}{b_0}\log\left(\frac{b}{b_1}\right)+\frac{R_0^2-b_0^2}{b_0}\log\left(\frac{b_1-b_0}{b-b_0}\right)\right]. \tag{85}$$

For b nearly equal to b_0, (85) becomes

$$x \approx \frac{9R_0^2\mu(R_0^2-b_0^2)}{a^4\rho U_0}\log\left(\frac{b_1-b_0}{b-b_0}\right). \tag{86}$$

Upon solving (86) for b and introducing the length x_0 we obtain

$$b-b_0 = (b_1-b_0)\exp\left[-x/x_0\right]. \tag{87}$$

Here x_0 is defined by

$$x_0 = \frac{9R_0^4\mu(1-b_0^2R_0^{-2})}{a^4\rho U_0}. \tag{88}$$

The concentration of particles $c(b,x)$ in the steady state is easily found, from the conservation equation, to be given by

$$c(b,x) = \frac{b_1(db_1/dt)\,c(b_1,0)}{b(db/dt)}. \tag{89}$$

By using (83) in (89), setting $c(b_1,0) = c_1$ and then using (87) to eliminate b_1 we obtain

$$\begin{aligned} c(b,x) &= \frac{b_1^2(b_1-b_0)}{b^2(b-b_0)}c_1 \\ &\approx \{b_0+(b-b_0)\exp\left[x/x_0\right]\}^2\frac{\exp\left[x/x_0\right]}{b^2}c_1, \\ &(b_0(1-\exp\left[-x/x_0\right]) < b < b_0+(R_0-b_0)\exp\left[-x/x_0\right]). \end{aligned} \tag{90}$$

Outside the range determined by the extreme initial radii $b_1 = 0$ and $b_1 = R_0$, which is given by the inequalities in (90), the concentration is zero. The concentration measured by Segre & Silberberg is in qualitative agreement with (90). Furthermore, they have found that their concentration data depend on x only through the ratio x/x_0, where x_0 is given by (88). This agreement shows that the force is of the order of magnitude given by (82). Since this force is of the first order in the Reynolds number, the actual force is also of this order and presumably would be obtained by completing the calculation of the first-order Stokes approximation to which we referred above.

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Appendix: on the solution of the Oseen equations

We wish to prove that every solution $\mathbf{W}$ of the reduced Oseen equations (46) and (47) can be represented in the form (48) in terms of two scalar functions ψ and χ satisfying (49) and (50). To do this we first rewrite (48) in component form after setting $\mathbf{W} = (W_1, W_2, W_3)$ and obtain

$$(W_1, W_2, W_3) = (\chi_{yy} + \chi_{zz}, -\psi_z - \chi_{xy}, \psi_y - \chi_{xz}). \tag{A 1}$$

We shall show that we can determine χ and ψ to satisfy (A 1). First we note that if χ satisfies (50) then $\chi_{yy} + \chi_{zz} = \chi_x - \chi_{xx}$. Then the first component of (A 1) becomes an ordinary differential equation for χ which has the solution

$$\chi = -e^x \int_0^x e^{-\xi} \int_0^\xi W_1(\eta, y, z)\, d\eta\, d\xi + a(y,z)\, e^x + b(y,z). \tag{A 2}$$

Here a and b are two 'constants' of integration.

To determine a and b we apply $\Delta_2 = \partial^2/\partial y^2 + \partial^2/\partial z^2$ to (A 2). From (A 1) we see that $\Delta_2 \chi = W_1$ while (46) shows that $\Delta_2 W_1 = W_{1x} - W_{1xx}$. Thus (A 2) yields

$$W_1 = e^x \int_0^x e^{-\xi} \int_0^\xi (W_{1\eta\eta} - W_{1\eta})\, d\eta\, d\xi + e^x \Delta_2 a + \Delta_2 b. \tag{A 3}$$

By evaluating the integral in (A 3) we obtain

$$W_1 = W_1 + e^x[\Delta_2 a(y,z) - W_{1x}(0,y,z)] + \Delta_2 b(y,z) + W_{1x}(0,y,z) - W_1(0,y,z). \tag{A 4}$$

In order that (A 4) hold for all values of x, y, and z it is necessary that a and b satisfy

$$\Delta_2 a(y,z) = W_{1x}(0,y,z) \tag{A 5}$$

and

$$\Delta_2 b(y,z) = W_1(0,y,z) - W_{1x}(0,y,z). \tag{A 6}$$

These equations have solutions which are determined up to additive harmonic functions. Thus (A 2) determines a class of functions χ which satisfy the first equation of (A 1).

Let us now use the second and third components of (A 1) to determine ψ. These components yield $\psi_z = -W_2 - \chi_{xy}$ and $\psi_y = W_3 + \chi_{xz}$. The integrability condition $\psi_{yz} - \psi_{zy} = 0$ is then

$$W_{3z} + \chi_{xzz} + W_{2y} + \chi_{xyy} = 0. \tag{A 7}$$

Since $\chi_{yy}+\chi_{zz} = W_1$ by (A 1), (A 7) becomes $W_{1x}+W_{2y}+W_{3z} = 0$ and this equation is satisfied according to (47). Thus we may compute ψ from the relation

$$\begin{aligned}\psi(x,y,z) &= \psi(x,0,0)+\int_0^y \psi_\eta(x,\eta,0)\,d\eta+\int_0^z \psi_\zeta(x,y,\zeta)\,d\zeta \\ &= \psi(x,0,0)+\int_0^y [W_3(x,\eta,0)+\chi_{xz}(x,\eta,0)]\,d\eta \\ &\quad -\int_0^z [W_2(x,y,\zeta)+\chi_{xy}(x,y,\zeta)]\,d\zeta. \qquad (\text{A } 8)\end{aligned}$$

We have now determined χ and ψ that satisfy (A 1). We have also verified that χ satisfies (50) since we have shown that $\chi_{yy}+\chi_{zz} = W_1$ and $\chi_x-\chi_{xx} = W_1$ so $\Delta\chi-\chi_x = 0$. Therefore we need only verify that ψ satisfies (49). To do so we first observe that $\psi_{zz} = (-W_2-\chi_{yx})_z$ and $\psi_{yy} = (W_3+\chi_{xz})_y$. Thus $\psi_{yy}+\psi_{zz} = W_{3y}-W_{2z}$. Next, from (A 8) we have

$$\begin{aligned}\psi_x-\psi_{xx} &= \psi_x(x,0,0)-\psi_{xx}(x,0,0)+\int_0^y [W_{3x}-W_{3xx}+\chi_{xxz}-\chi_{xxxz}]\,d\eta \\ &\quad -\int_0^z [W_{2x}-W_{2xx}+\chi_{xxy}-\chi_{xxxy}]\,d\zeta. \qquad (\text{A } 9)\end{aligned}$$

From (46), $W_{3x}-W_{3xx} = \Delta_2 W_3$ while from (50) and (A 1) $\chi_{xxz}-\chi_{xxxz} = \Delta_2\chi_{xz} = W_{1xz}$. Similarly, $W_{2x}-W_{2xx} = \Delta_2 W_2$ and $\chi_{xxy}-\chi_{xxxy} = W_{1xy}$. Furthermore, from (47) $W_{1xz} = -(W_{2y}+W_{3z})_z$ and $W_{1xy} = -(W_{2y}+W_{3z})_y$. Then (A 9) becomes

$$\begin{aligned}\psi_x-\psi_{xx} &= \psi_x(x,0,0)-\psi_{xx}(x,0,0)+\int_0^y [W_{3\eta\eta}-W_{2\eta z}]\,d\eta-\int_0^z [W_{2\zeta\zeta}-W_{3y\zeta}]\,d\zeta \\ &= \psi_x(x,0,0)-\psi_{xx}(x,0,0)-W_{3y}(x,0,0)+W_{2z}(x,0,0) \\ &\quad +W_{3y}(x,y,z)-W_{2z}(x,y,z). \qquad (\text{A } 10)\end{aligned}$$

Let us now require that $\psi(x,0,0)$, which is so far arbitrary, satisfy the equation

$$\psi_x(x,0,0)-\psi_{xx}(x,0,0) = W_{3y}(x,0,0)-W_{2z}(x,0,0). \qquad (\text{A } 11)$$

Then (A 10) shows that $\psi_x-\psi_{xx} = W_{3y}-W_{2z}$ which we saw above is equal to $\psi_{yy}+\psi_{zz}$. Thus ψ satisfies (49) and the theorem is proved.

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